Formative solutions for TMA04 202xJ

TMA 04

Solution to Question 7 – 9 marks

(a) By Definition 2.1 of Chapter 17 (see also Worked Exercises 2.3 and 2.4 therein), it is enough to show that A is closed under subtraction of elements of A and under multiplication by every element of R.

Closure under subtraction: Let $\alpha = a + b\sqrt{-13}$ and $\beta = c + d\sqrt{-13}$ be elements of A, that is, $a - b \equiv 0 \pmod 7$ and $c - d \equiv 0 \pmod 7$. Then we also have $(a - c) - (b - d) = (a - b) - (c - d) \equiv 0 \pmod 7$ and so $\alpha - \beta = (a - c) + (b - d)\sqrt{-13} \in A$.

Closure under multiplication: Let $\alpha = a + b\sqrt{-13}$ be an element of A and so $a - b \equiv 0 \pmod 7$. If $\eta = x + y\sqrt{-13}$ is an arbitrary element of R, then

$$\eta\alpha = (ax - 13by) + (ay + bx)\sqrt{-13}.$$

To see that $\eta\alpha \in A$, one needs to show that

$$(ax - 13by) - (ay + bx) \equiv 0 \pmod 7.$$

This follows by observing that

$$(ax - 13by) - (ay + bx) = x(a - b) - y(a + 13b)$$

but $a + 13b = a + (14 - 1)b \equiv a - b \equiv 0 \pmod 7$ and so

$$x(a - b) - y(a + 13b) \equiv x(a - b) \equiv 0 \pmod 7,$$

as required.

Alternatively: from $ax - 13by = ax + by - 14by \equiv ax + by \pmod 7$ one has $(ax - 13by) - (ay + bx) \equiv (ax + by) - (ay + bx) = (a - b)(x - y) \equiv 0 \pmod 7$, since $a - b \equiv 0 \pmod 7$.

Worked Exercises 2.3 and 2.4 of Chapter 17

(b) To show that $AB = \langle 7 \rangle$, we show separately that $AB \subseteq \langle 7 \rangle$ and $\langle 7 \rangle \subseteq AB$.

The calculations are similar to those in Worked Exercise 3.10 of Chapter 17.

The ideal AB consists of (finite) sums of products of the form $(a + b\sqrt{-13})(c + d\sqrt{-13})$, where $a, b, c, d \in \mathbb{Z}$ satisfy $a - b \equiv 0 \pmod 7$ and $c + d \equiv 0 \pmod 7$, or, equivalently, $a \equiv b \pmod 7$ and $c \equiv -d \pmod 7$.

Evaluating the product gives

$$(a + b\sqrt{-13})(c + d\sqrt{-13}) = (ac - 13bd) + (ad + bc)\sqrt{-13}$$

and the aim is to show that both bracketed terms are multiples of 7.

Multiplying $a \equiv b \pmod 7$ with $c \equiv -d \pmod 7$ gives $ac \equiv -bd \pmod 7$ and so $0 \equiv ac + bd \equiv ac - 13bd \pmod 7$.

Similarly, by multiplying $a \equiv b \pmod 7$ with $d \equiv -c \pmod 7$, we obtain $ad \equiv -bc \pmod 7$ and so $ad + bc \equiv 0 \pmod 7$.

It follows that both $ac - 13bd$ and $ad + bc$ are multiples of 7 and hence $AB \subseteq 7R = \langle 7 \rangle$.

To prove the reverse inclusion is a little trickier: we need to show that 7 is in AB, so we need to find a way of writing 7 as a finite sum of elements that we can show are in AB.

We first notice that 7 lies in both A and B and hence $7 \times 7 = 49 \in AB$.

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We now observe that $1+\sqrt{-13} \in A$ and $1-\sqrt{-13} \in B$ and so
$14 = (1+\sqrt{-13})(1-\sqrt{-13}) \in AB$.

But then $49 - 3 \times 14 = 7$ is also in AB, as required and $\langle 7 \rangle \subseteq AB$.

We conclude that $AB = \langle 7 \rangle$.

Worked Exercise 3.10 of Chapter 17

Solution to Question 8

Throughout this question, we are using Definition 2.6 of Chapter 14, HB p72.

(a) For the metric $d((a_1,a_2),(b_1,b_2)) = 6|b_1-a_1| + |b_2-a_2|$, we find:

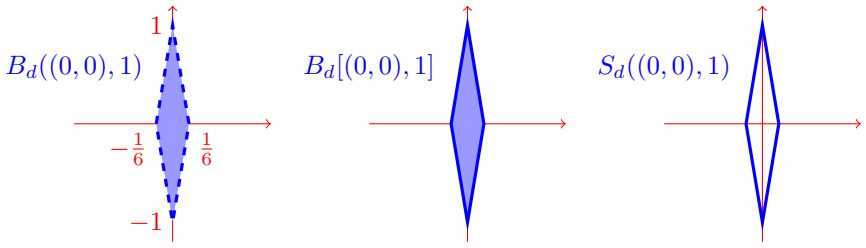
$$B_d((0,0),1) = \{(x,y) \in \mathbb{R}^2 : 6|x| + |y| < 1\},$$

$$B_d[(0,0),1] = \{(x,y) \in \mathbb{R}^2 : 6|x| + |y| \leq 1\},$$

and

$$S_d((0,0),1) = \{(x,y) \in \mathbb{R}^2 : 6|x| + |y| = 1\}.$$

Hence a sketch of the three sets is as below.



(b) $d((a_1,a_2),(b_1,b_2)) = (|b_1-a_1|^4 + |b_2-a_2|^4)^{1/4}$ For this choice of metric,

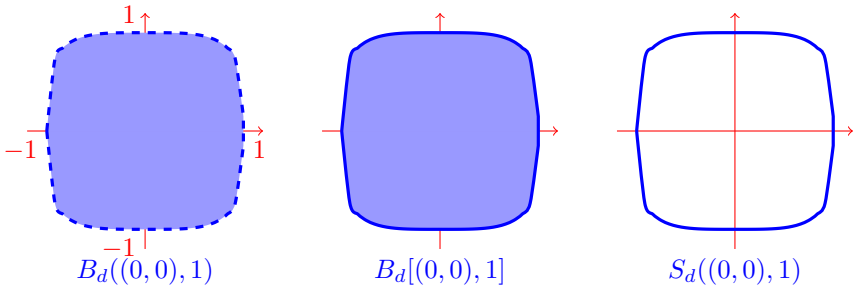
$$B_d((0,0),1) = \{(x,y) \in \mathbb{R}^2 : |x|^4 + |y|^4 < 1\},$$

$$B_d[(0,0),1] = \{(x,y) \in \mathbb{R}^2 : |x|^4 + |y|^4 \leq 1\},$$

and

$$S_d((0,0),1) = \{(x,y) \in \mathbb{R}^2 : |x|^4 + |y|^4 = 1\}.$$

Hence an approximate sketch of the three sets is as below.



This metric is in fact the product metric e_4 on $\mathbb{R} \times \mathbb{R}$ obtained from the Euclidean metric on $\mathbb{R}$ — see the remark in Chapter 15, page 122.

(c) $d((a_1,a_2),(b_1,b_2)) = d_0(a_1,b_1) + 6|b_2-a_2|$

This metric is similar to the 'mixed' metric described on page 93 of Book D.

For this metric we use the fact that $d((a_1,a_2),(b_1,b_2)) \geq 1$ unless $a_1 = b_1$.

$$\begin{aligned} B_d((0,0),1) &= \{(x,y) \in \mathbb{R}^2 : d_0(x,0) + 6|y| < 1\} \\ &= \{(0,y) \in \mathbb{R}^2 : 6|y| < 1\} \\ &= \{(0,y) \in \mathbb{R}^2 : -1/6 < y < 1/6\} \end{aligned}$$

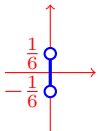
For the closed ball,

$$\begin{aligned} B_d[(0,0),1] &= \{(x,y) \in \mathbb{R}^2 : d_0(x,0) + 6|y| \leq 1\} \\ &= \{(0,y) : 6|y| \leq 1\} \cup \{(x,y) \in \mathbb{R}^2 : x \neq 0,\ 1 + 6|y| = 1\} \\ &= \{(0,y) : -1/6 \leq y \leq 1/6\} \cup \{(x,0) : x \in \mathbb{R},\ x \neq 0\} \\ &= \{(0,y) : -1/6 \leq y \leq 1/6\} \cup \{(x,0) : x \in \mathbb{R}\} \end{aligned}$$

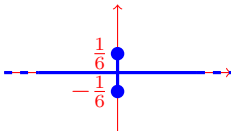
And for the unit sphere

$$\begin{aligned} S_d((0,0),1) &= \{(x,y) \in \mathbb{R}^2 : d_0(x,0) + 6|y| = 1\} \\ &= \{(0,y) : 6|y| = 1\} \cup \{(x,y) \in \mathbb{R}^2 : x \neq 0,\ 1 + 6|y| = 1\} \\ &= \{(0,1/6),\ (0,-1/6)\} \cup \{(x,0) : x \in \mathbb{R},\ x \neq 0\} \end{aligned}$$

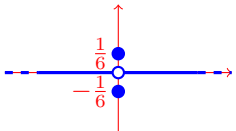
Hence a sketch of the three sets is as below.



$B_d((0,0),1)$



$B_d[(0,0),1]$



$S_d((0,0),1)$

See Section 2.3 of Chapter 14.

Solution to Question 9

(a) $f_5(1) = 5^{-0} = 1$, $f_5(5) = \frac{1}{5}$ and $f_5(5^n) = 5^{-n}$.

(b) (i) Certainly $f_5(x) \geq 0$ for all $x \in \mathbb{Q}$ and $f_5(0) = 0$ by the definition.
So it only remains to show that $f_5(x) = 0$ implies $x = 0$. But if $x \neq 0$, then $f_5(x) = 5^{-m}$ for some $m \in \mathbb{Z}$ and this is not equal to zero. So $f_5(x) = 0$ implies $x = 0$.

(ii) If $x = 0$, then there is nothing to prove.
So suppose $x \neq 0$ and $x = 5^m a/b$ with 5 not a divisor of a and b, then $-x = 5^m(-a)/b$ and 5 does not divide $-a$ and b.
Hence $f_5(x) = f_5(-x)$.

(iii) If either of x or y is equal to zero, then the result is immediate.
So suppose both x and y are not zero and that $r, s \in \mathbb{Z}$ are such that $x = 5^r a/b$ and $y = 5^s c/d$ with 5 not dividing a, b, c, d.
Then $f_5(x) = 5^{-r}$ and $f_5(y) = 5^{-s}$, and so

$$\max\{f_5(x),\, f_5(y)\} = \max\{5^{-r}, 5^{-s}\} = 5^{-\min\{r,s\}}.$$

Now

$$x + y = 5^r \frac{a}{b} + 5^s \frac{c}{d} = \frac{5^r ad + 5^s bc}{bd} \text{ and } 5 \not| bd.$$

Thus if $m = \min\{r, s\}$, then $x + y = 5^m \frac{5^{r-m}ad + 5^{s-m}bc}{bd}$ and (at least) one of 5^{r-m} and 5^{s-m} is equal to one. It is tempting to conclude here that 5 does not divide $5^{r-m}ad + 5^{s-m}bc$ but this is wrong: consider $5^0 \times 1 \times 3 + 5^0 \times 1 \times 2$.
Now $5^{r-m}ad + 5^{s-m}bc = 5^A k$ for some $A \in \{0, 1, 2, \ldots\}$ and $k \in \mathbb{Z}$ that is coprime to 5.
Hence $x + y = 5^m 5^A \frac{k}{bd}$ and 5 is coprime to both k and bd. Hence $f_5(x + y) = 5^{-m}5^{-A} \leq 5^{-m}$, since $A \geq 0$, as required.

(c) Using the result of (b), it is a small step to verifying that d_5 satisfies (M1)–(M3) of the definition of a metric Definition 2.1 of Chapter 14, HB p72.

(i) (M1) Certainly (b)(i) immediately implies that (M1) holds.

(ii) (M2) Using (b)(ii), we see that

$$d_5(r, s) = f_5(r - s) = f_5(-(r - s)) = f_5(s - r) = d_5(s, r),$$

meaning that (M2) holds.

(iii) (M3) Finally, if $r, s, t \in \mathbb{Q}$, then using (b)(iii)

$$\begin{aligned} d_5(r, t) &= f_5(r - t) = f_5((r - s) + (s - t)) \\ &\leq \max\{f_5(r - s),\, f_5(s - t)\} \\ &= \max\{d_5(r, s),\, d_5(s, t)\} \\ &\leq f_5(r - s) + f_5(s - t) \\ &= d_5(r, s) + d_5(s, t) \end{aligned}$$

and so (M3) holds.

If a metric d for a space X satisfies $d(x, y) \leq \max\{d(x, z), d(z, x)\}$ for each $x, y, z \in X$, then it is called an *ultrametric*. Thus you have just shown that the 5-adic metric is an ultrametric. Other examples of ultrametric spaces that you have met include Cantor space (with the Cantor metric) and any set when given the discrete metric. Note that the proof that d_5 is a metric involved analysing the properties of f_5 in some detail: we aren't using general results or theorems. This is usually the case when looking at a new distance function: you have to get your hands dirty!

All of the examples of metric spaces given in Chapter 15 are of some relevance but it may be worth pointing out in particular, the Cantor metric, which is Section 2 of Chapter 15.

Solution to Question 10

(a) (i) **(M1)** Clearly $d_G(g,h) \geq 0$ and

$$\begin{aligned} d_G(g,h) = 0 &\text{ if, and only if, } gh^{-1} \in H_0 \\ &\text{if, and only if, } gh^{-1} = e \\ &\text{if, and only if, } g = h. \end{aligned}$$

(ii) **(M2)** $d_G(g,h) = m$ if, and only if,
$gh^{-1} \in H_m$ (and m is the smallest possible such number).
But, by Proposition 2.2 of Chapter 5, HB page 25,

$$\begin{aligned} gh^{-1} \in H_m &\text{ if, and only if, } (gh^{-1})^{-1} = hg^{-1} \in H_m \\ &\text{if, and only if, } d_G(h,g) = m. \end{aligned}$$

(iii) **(M3)**
Let $g, k, h \in G$ be given.
Suppose $d_G(g,h) = m$ and consider $d_G(g,k) + d_G(k,h)$.
Let $d_G(g,k) = n_1$ and $d_G(k,h) = n_2$, where n_1 and n_2 are non-negative integers.
We need to show that $m \leq n_1 + n_2$.
Since $d_G(g,h) \geq 0$ for all $g, h \in G$ if $n_i \geq m$ for $i = 1$ or 2 then we are done.
Therefore assume, for a contradiction, that n_1 and n_2 are both less than m.
Without loss of generality, we may assume that $n_1 \leq n_2 < m$.
Then $gk^{-1} \in H_{n_1} \leq H_{n_2}$ and $kh^{-1} \in H_{n_2}$.
Therefore $gk^{-1}kh^{-1} = gh^{-1} \in H_{n_2}$ contradicting the assumption that $n_2 < m$. In fact, $m = \max\{n_1, n_2\}$.

(b) The definition of *eventually constant* is given in
Definition 3.4 of Chapter 14, HB page 73.

Assume first that the sequence (g_i) is eventually constant. Then there is an $N \in \mathbb{N}$ and a $g \in G$ such that $g_i = g$ for all $i > N$. Therefore for all $i > N, d_G(g_i, g) = 0$ and so the sequence $(d_G(g_i, g))$ is null and (g_i) is d_G-convergent to g.

(Alternatively this follows from Worked Exercise 3.5 of Chapter 14.)

Assume now that the sequence (g_i) is d_G-convergent to g. Then the sequence $(d_G(g_i, g))$ is null so in particular there must be an $N \in \mathbb{N}$ such that for $i > N, d_G(g_i, g) < 1$. But this means that for $i > N$, $d_G(g_i, g) = 0$ and so for $i > N$, $g_i = g$. That is (g_i) is eventually constant (and takes the value g).

All of the examples of metric spaces given in Chapter 15 are of some relevance.

For part (b) Worked Exercise 3.5 and Exercise 3.2 of Chapter 14 are helpful.